Adaptivity does not help: Nearly tight lower bounds for Boolean monotonicity testing

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Adaptivity does not help: Nearly tight lower bounds for Boolean monotonicity testing
… or *How to hide anti-dictators*

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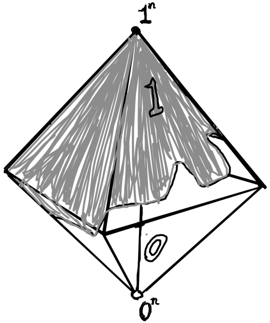
Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$:



monotone if $f(x) \leq f(y)$ for all $x, y \in \{0,1\}^n$ with $x \preceq y$.

Given black-box access to an $f : \{0,1\}^n \to \{0,1\}$ that is not monotone, how many queries does it take to find a violating pair:

$$(x,y) \text{ with } x \preceq y \text{ but } f(x) = 1 \text{ and } f(y) = 0$$

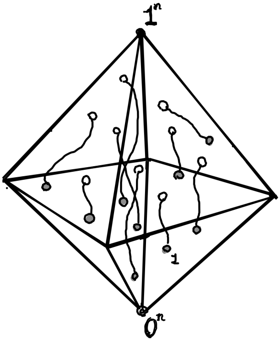
Given black-box access to an $f : \{0,1\}^n \to \{0,1\}$ that is far from monotone, how many queries does it take to find a violating pair:

$$(x, y) \text{ with } x \preceq y \text{ but } f(x) = 1 \text{ and } f(y) = 0$$

far: at least 1% *of entries flipped to become monotone*

one-sided monotonicity testing

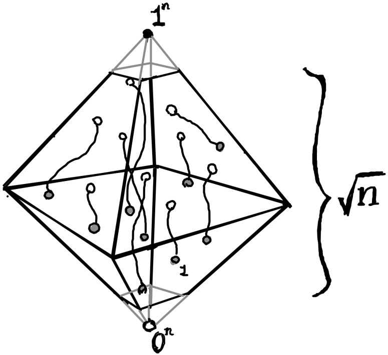
Far-from-monotone Functions



Equivalent Characterization [Fischer et al. 02]

f is far from monotone iff there exist $\Omega(2^n)$ disjoint violating pairs.

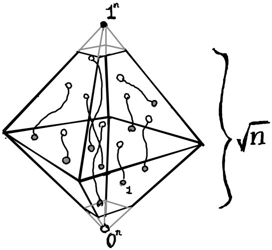
Middle layers



Equivalent Characterization [Fischer et al. 02]

f is far from monotone iff there exist $\Omega(2^n)$ disjoint violating pairs.

Given f : middle layers $\to \{0,1\}$ with $\Omega(2^n)$ disjoint violating pairs:



How many queries are needed to find a violating pair?

Property testing

Monotonicity testing:

- [Goldreich, Goldwasser, Lehman, Ron and Samordinsky 98]
- [DGL+99, FLN+02, HK08, BCGM12, RRS+12, BBM12, BRY13, CS13, CS14, CST14, KMS15, CDST15, BB16, CWX 17, CS 19]
- More papers on real-valued functions over the hypergrid [HK07, AC06, SS08, FR10, BGJ+12, CS13, BRY14a, BRY14b, BCS18, BCS20, BKR24, HY20, BKKM22, BCS23, BCS25].

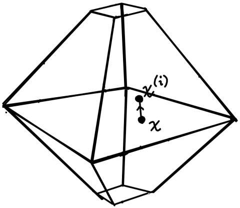
This Talk

Theorem (CCCPS 25)

Any adaptive algorithm requires $n^{0.5-c}$ queries, for any constant $c > 0$.

Nearly closes the gap of $O(\sqrt{n})$ vs $\Omega(n^{1/3})$
Works against adaptive, two-sided algorithms.

Edge tester



Pick a random edge $(x, x^{(i)})$ and query both points.

Theorem (Goldreich et al. 98)

If f is far from monotone, then

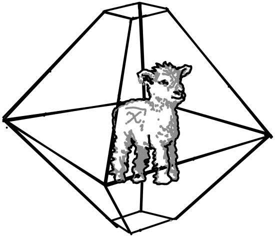
$$\mathbf{E}_{\mathbf{x}\sim\{0,1\}^n}\big[\textit{num of violating edges adj. to } \mathbf{x}\big] = \Omega(1).$$

So $\Omega(2^n)$ violating edges and $O(n)$ queries suffice.

Directed analogue of a folklore isoperimetric inequality:

$$\mathbf{E}_{\mathbf{x}\sim\{0,1\}^n}\big[\text{num of bichromatic edges adj. to } \mathbf{x}\big] = \Omega\big(\text{var}(f)\big).$$

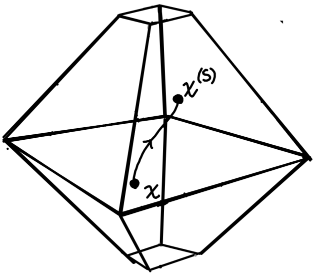
Anti-dictatorship functions



$\Omega(n)$ queries are required

~~Left to the audience as an exercise~~

Pair tester



Let ℓ be a parameter. Draw a random point x, flip a set S of ℓ random 0's of x to 1, and query both points x and $x^{(S)}$.

Theorem (Khot, Minzer and Safra 15)

If f is far from monotone, then

$$\mathbf{E}_{\mathbf{x}\sim\{0,1\}^n}\left[\sqrt{\textit{num of violating edges adj. to } \mathbf{x}}\right] = \Omega(1),$$

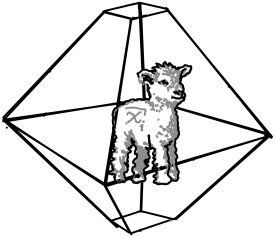
which led to an $O(\sqrt{n})$ upper bound for the path tester.

Directed analogue of an isoperimetric inequality of [Talagrand 93]:

$$\mathbf{E}_{\mathbf{x}\sim\{0,1\}^n}\left[\sqrt{\text{num of violating edges adj. to } \mathbf{x}}\right] = \Omega\big(\mathrm{var}(f)\big).$$

- [Chakrabarty and Seshadhri 13]: $O(n^{7/8})$
- [Chen, Servedio and Tan 14]: $O(n^{5/6})$

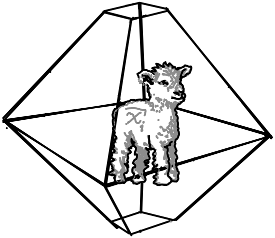
Nonadaptive lower bounds



Theorem (Fischer, Lehman, Newman, Raskhodnikova, Rubinfeld and Samorodnitsky 02)

Any nonadaptive algorithm that finds a violating pair needs $\sqrt{n}$ queries.

But not against adaptive enemies ...



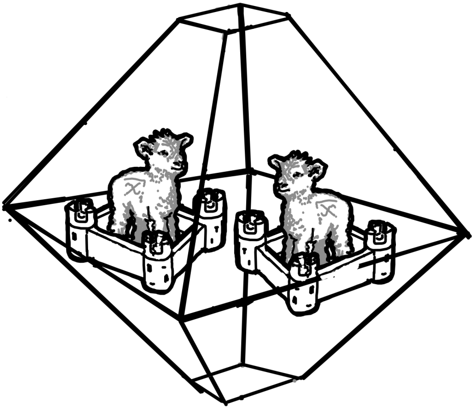
Binary search or variance estimation

Adaptive Lower Bounds

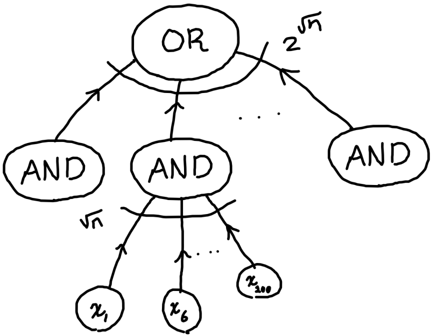
Adaptive lower bounds, or ... how to hide anti-dictators:

1. Talagrand Functions [Belovs and Blais 16]: $\Omega(n^{1/4})$
2. Two-level Talagrand Functions [C, Waingarten and Xie 17]: $\Omega(n^{1/3})$
3. This work: *Multi-level Talagrand Functions*

Blueprint of [Belovs and Blais 16]



Talagrand's function



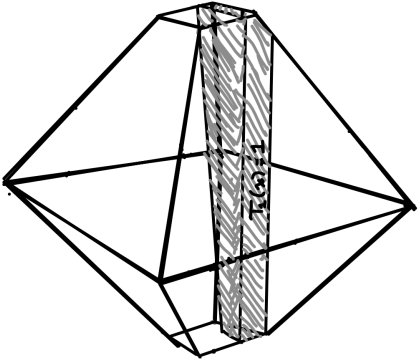
Let $s = \sqrt{n}$. OR of 2^s terms: $T_1, \ldots, T_{2^s}$

Each term T_i: s variables, sampled independently and uniformly.

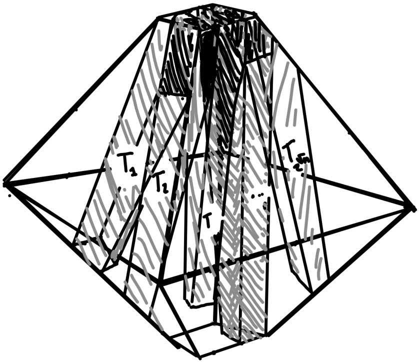
Why setting $s = \sqrt{n}$?

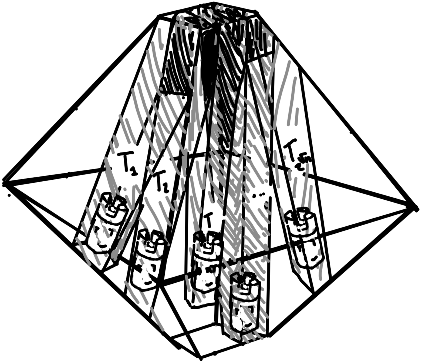
Largest possible to make the function interesting: Fixing a $z \in \{0,1\}^n$

$$\Pr_{T}\left[T(z)=1\right] = \left(\frac{|z|}{n}\right)^{\sqrt{n}} = \left(\frac{(n/2) \pm \sqrt{n}}{n}\right)^{\sqrt{n}} = \left(1 \pm \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} \cdot \frac{1}{2^{\sqrt{n}}} = \Theta(1) \cdot \frac{1}{2^{\sqrt{n}}}.$$



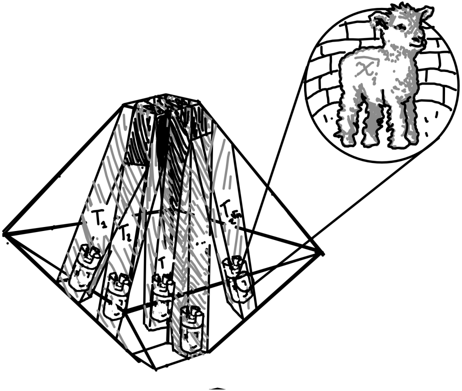
T1(x)=1


T_1
T_2
T
$T_{2^{\sqrt{n}}}$



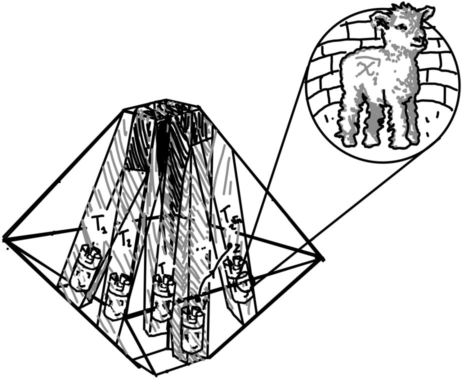
The castles!

Blueprint of [Belovs and Blais 16]



1. The function is far from monotone; and
2. The only violations are (x, y) that (1) lie in the same castle and (2) do not agree on the anti-dictator of this castle.

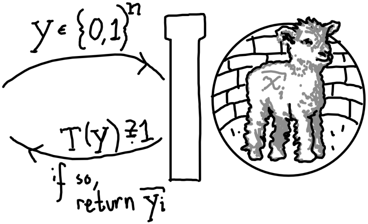
Blueprint of [Belovs and Blais 16]



1. The function is far from monotone; and
2. The only violations are (x, y) that (1) lie in the same castle and (2) do not agree on the anti-dictator of this castle.

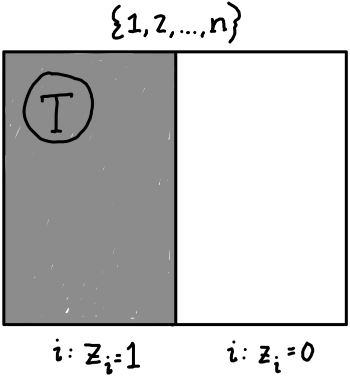
The castle game

Given a point z with $T(z) = 1$ for free:

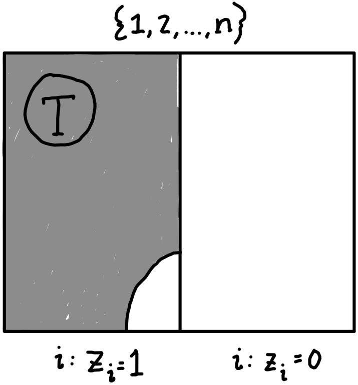


Goal: Find y, y' with $T(y) = T(y') = 1$ and $y_i \neq y'_i$.

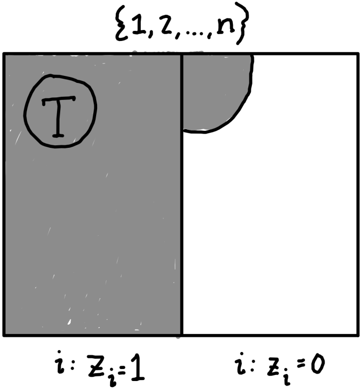
$\Omega(\sqrt{n})$?



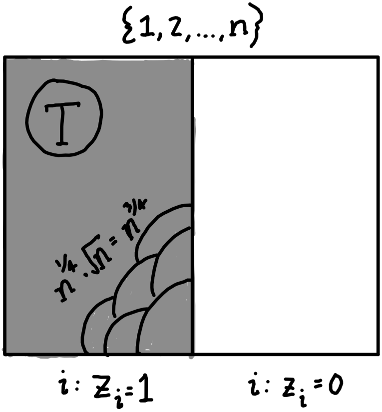
$\Omega(\sqrt{n})$?



$\Omega(\sqrt{n})$?

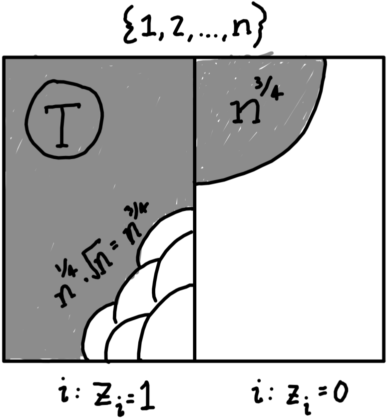


Quadratic speedup



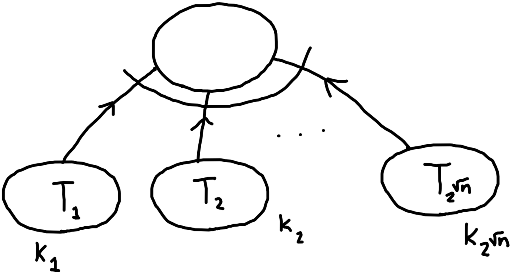
Stage 1: Use $n^{1/4}$ queries to find $\Omega(n^{3/4})$ variables not in T.

Quadratic speedup

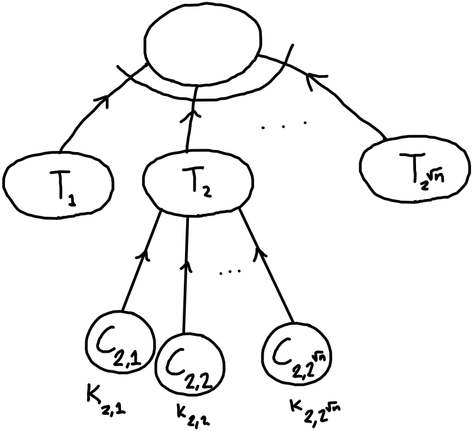


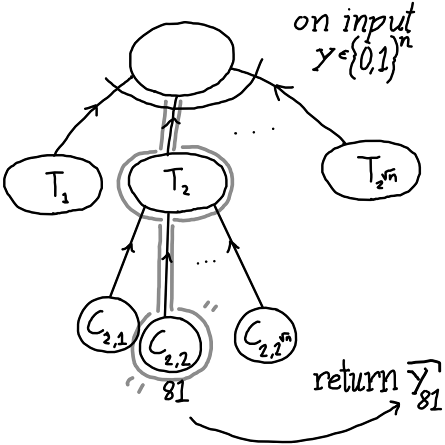
Stage 2: Use $n^{1/4}$ queries to search through the white region.

Construction of [Chen, Waingarten and Xie 17]



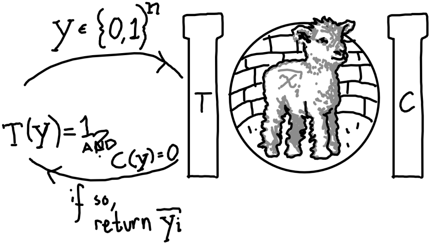
Construction of [Chen, Waingarten and Xie 17]




on input
$y \in \{0,1\}^n$
T_1
T_2
$T_{2^{\sqrt{n}}}$
$C_{2,1}$
$C_{2,2}$
$C_{2,2^{\sqrt{n}}}$
81
return $\overline{y_{81}}$

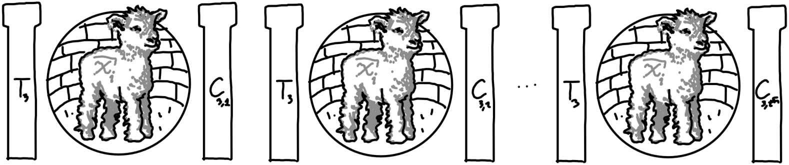
The new castle game

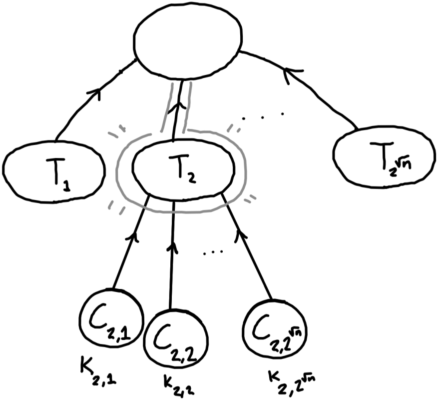
Given a point z with $T(z) = 1$ and $C(y) = 0$ for free:

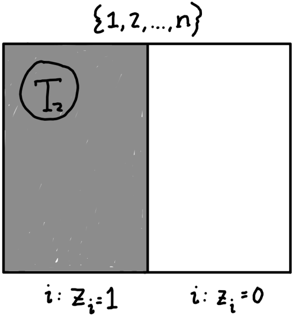


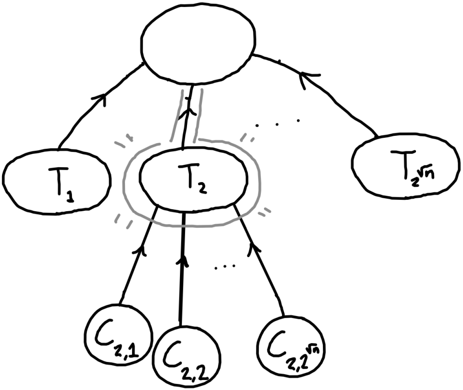
Goal: Find y, y' with $T(y) = T(y') = 1$, $C(y) = C(y') = 0$ and $y_i \neq y'_i$.

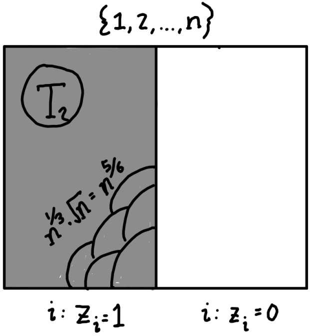
Quadratic speedup strikes back ...

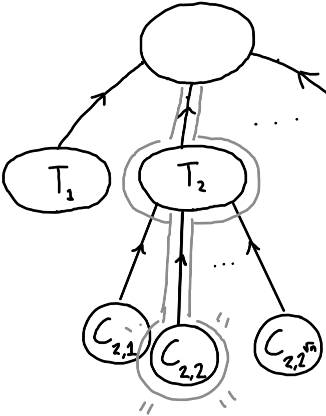



T_1
T_2
$T_{2^{\sqrt{n}}}$
$C_{2,1}$
$C_{2,2}$
$C_{2,2^{\sqrt{n}}}$
$K_{2,1}$
$K_{2,2}$
$K_{2,2^{\sqrt{n}}}$

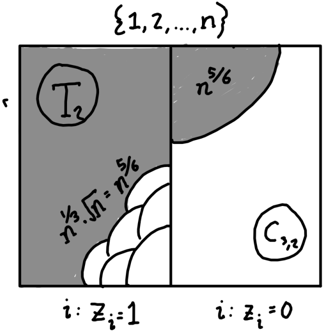

$\{1, 2, \dots, n\}$
T_2
$i: z_i = 1$
$i: z_i = 0$


T1
T2
T2√n
C2,1
C2,2
C2,2√n

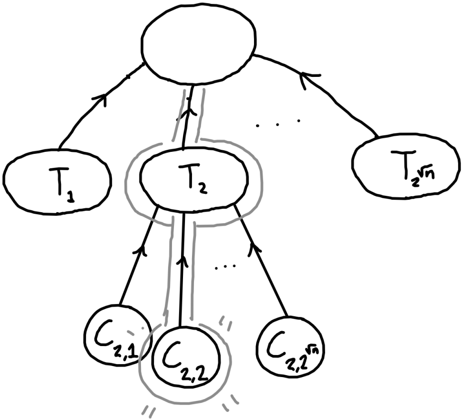

{1, 2, …, n}
T2
n1/3 · √n = n5/6
i: zi = 1
i: zi = 0

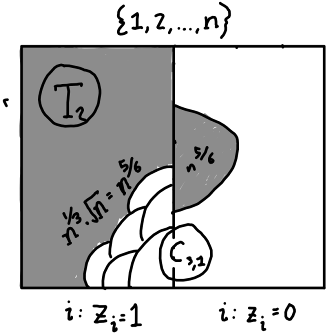


T_1
T_2
$T_{2\sqrt{n}}$
$C_{2,1}$
$C_{2,2}$
$C_{2,2\sqrt{n}}$



$\{1, 2, \ldots, n\}$
T_2
$n^{5/6}$
$n^{1/3} \cdot \sqrt{n} = n^{5/6}$
$C_{3,2}$
$i : z_i = 1$
$i : z_i = 0$


T
T
T
T
C
C
C
...
...


{1, 2, ..., n}
T
C
i : Z_i = 1
i : Z_i = 0

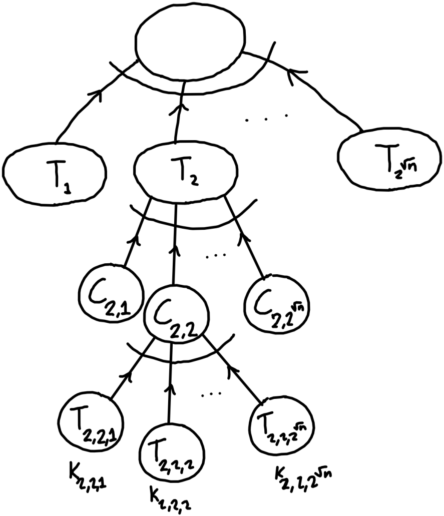
Summary

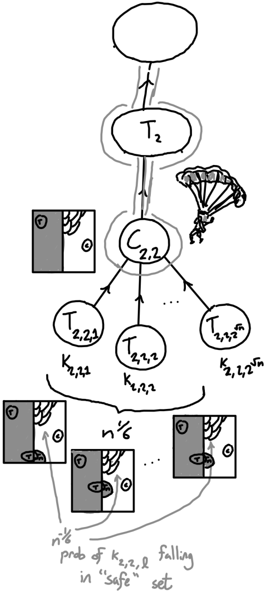
- one-level: $q^2\sqrt{n} \Rightarrow q = \Omega(n^{1/4})$
- two-level: $q^{3/2}\sqrt{n} \Rightarrow q = \Omega(n^{1/3})$
  - q queries to obtain $q\sqrt{n}$ safe variables for the term
  - for each clause below, $\sqrt{q}$ queries suffice to search through $q\sqrt{n}$
  - repeat for $\sqrt{q}$ clauses below to cover $q^{3/2}\sqrt{n}$ in total

More levels?

- one-level: $q^2\sqrt{n} \Rightarrow q = \Omega(n^{1/4})$
- two-level: $q^{3/2}\sqrt{n} \Rightarrow q = \Omega(n^{1/3})$
  - q queries to obtain $q\sqrt{n}$ safe variables for the term
  - for each clause below, $\sqrt{q}$ queries suffice to search through $q\sqrt{n}$
  - repeat for $\sqrt{q}$ clauses below to cover $q^{3/2}\sqrt{n}$ in total
- three-level: $q^{4/3}\sqrt{n} \Rightarrow q = \Omega(n^{3/8})$?

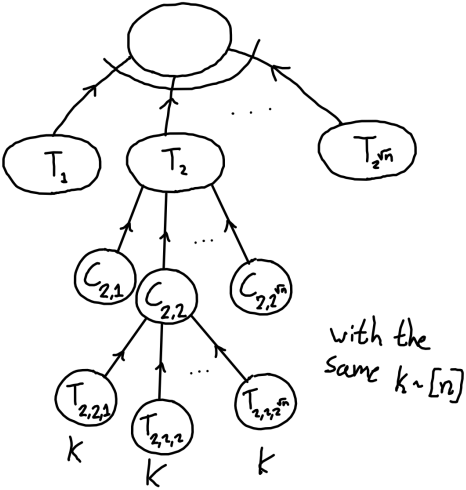
  $\ldots$
- ℓ-level: $q^{(\ell+1)/\ell}\sqrt{n} \Rightarrow q = \Omega(n^{\ell/2(\ell+1)})$?


T1
T2
T2√n
C2,1
C2,2
C2,2√n
T2,2,1
T2,2,2
T2,2,2√n
K2,2,1
K2,2,2
K2,2,2√n



T_2
$C_{2,2}$
$T_{2,2,1}$
$T_{2,2,2}$
$T_{2,2,2^{\sqrt{n}}}$
$K_{2,2,1}$
$K_{2,2,2}$
$K_{2,2,2^{\sqrt{n}}}$
$n^{1/6}$
$n^{-1/6}$
prob of $K_{2,2,\ell}$ falling in "safe" set

The fix after eight years ...



More levels!

- one-level: $q^2\sqrt{n} \Rightarrow q = \Omega(n^{1/4})$
- two-level: $q^{3/2}\sqrt{n} \Rightarrow q = \Omega(n^{1/3})$
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  - for each clause below, $\sqrt{q}$ queries suffice to search through $q\sqrt{n}$
  - repeat for $\sqrt{q}$ clauses below to cover $q^{3/2}\sqrt{n}$ in total
- three-level: $q^{4/3}\sqrt{n} \Rightarrow q = \Omega(n^{3/8})$

  $\ldots$
- ℓ-level: $q^{(\ell+1)/\ell}\sqrt{n} \Rightarrow q = \Omega(n^{\ell/2(\ell+1)})$

Open Problems

- Our lower bounds: $\Omega(n^{0.5-c})$ for any $c > 0$
- [Minzer 26]: $\Omega(n^{0.5-o(1)})$ by a distance amplification lemma
- Close the gap with $\tilde{\Omega}(\sqrt{n})$?
- Reduce the exponential gap between upper $\left(2^{\tilde{O}(\sqrt{n})}\right)$ and lower bounds $\left(2^{\Omega\left(n^{1/4}\right)}\right)$ for tolerant testing of monotonicity and many other related problems.
  In particular, can estimating distance from monotonicity be done faster than learning?

Thank you!

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William Pires
Jonah Stockwell*

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*To whom the art credit is due; an undergrad applying for a PhD position in the coming cycle!!!