Equivalences between TFNP^{dt} and Low-Level Proof Systems with an example in $\mathsf{PPA}^{dt} \cong \mathbb{F}_2$-Nullstellensatz

Presented by Mark Chen, Hao Cui and Jiaqian Li

Spring, 2025

Search Problems (Review)

Unrelativized total search problems in **NP**, TFNP

Let $\mathcal{O}$ be encoded by $[m(n)]$, where m is a polynomial.

Definition (Search problem)

A search problem is $S \subseteq \{0,1\}^n \times \mathcal{O}$, such that provided $x \in \{0,1\}^n$, we need to find one of the elements in

$$S(x) := \{o \in \mathcal{O} : (x, o) \in S\}.$$

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Definition (Total search problem)

A search problem $S \in \{0,1\}^n \times \mathcal{O}$ is *total* if $\forall x \in \{0,1\}^n$, $|S(x)| \geq 1$.

Definition (TFNP)

A total search problem $S \in \mathsf{TFNP}$ that is in **NP**, i.e. $\exists$a poly-time Turing machine $\mathcal{M}$ such that $\forall x \in \{0,1\}^n, \forall y \in S(x)$, $\mathcal{M}(x, y) = 1 \iff (x, y) \in S$.

Bridging $\mathsf{TFNP}^{\mathsf{dt}}$ with Low-level Proof Systems

Query total search problems in query **NP**, $\mathsf{TFNP}^{\mathsf{dt}}$

Definition ($\mathsf{TFNP}^{\mathsf{dt}}$)

A total search problem $S \subseteq \{0,1\}^N \times \mathcal{O}$ is in $\mathsf{TFNP}^{\mathsf{dt}}$ if $\forall y \in \mathcal{O}$, there exists a polylog$(N)$-depth decision tree T_y such that $\forall x \in \{0,1\}^N$,

$$T_y(x) = 1 \iff (x, y) \in S.$$

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Remark: One can think of x instance given as a truth table, that typically has $N \approx 2^n$, so polylog(N) = quasipoly(n)-depth trees are meant to capture the query analogue of being “poly-time computable.”

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Definition ($\mathsf{PPA} \subseteq \mathsf{TFNP}$)

A search problem $S \in \mathsf{PPA}$ iff it is a TFNP problem whose totality is guaranteed by "hand-shaking lemma."

Lemma (Hand-shaking)

The sum of the degrees of all vertices in a graph is equal to twice the number of edges (in particular, the sum must be even).

We take for granted that all total TFNP problems by "hand-shaking" is poly-time reducible to End-of-undirected-Line:

Definition (End-of-undirected-Line, Informal)

Given the succinct description (as a poly-sized circuit) of a very large degree-2 undirected acyclic graph (with exponentially many nodes). We are also given v_0 which has exactly one neighbor.

Find $v' \neq v_0$ which also has only one neighbor.

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Lemma

End-of-undirected-Line *is PPA-complete (by poly-time many-one reduction).*

The notion that $\mathsf{PPA} \subseteq \mathsf{TFNP}$ is the set of all TFNP problems that are poly-time many-one reducible to End-of-undirected-Line is helpful.

Unrelativized to query model, roughly speaking, replace the "efficiently reducible" with "reducible with a family of low-height trees."

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Next up, we will (1) define a query analogue of a $\mathsf{PPA}^{\mathsf{dt}}$-complete problem and (2) specify how "low-depth" decision trees reductions, $\{\mathcal{T}_v\}_{v\in V}$, define problems contained in $\mathsf{PPA}^{\mathsf{dt}}$.

Definition ($\mathsf{PPA}^{\mathsf{dt}}$-OddDeg)

Fix an input $x \in \{0,1\}^N$. An *instance* is a tuple

$$(V, v^* \in V, \{T_v\}_{v \in V}), \qquad |V| \le \mathrm{poly}(N),$$

where

- each T_v is a polylog(N)-height *deterministic decision tree* that, on x, outputs a list $T_v(x) \subseteq V$ with $|T_v(x)| \le 2$;
- the (implicit) graph is:

$$G_x = (V, E_x), \quad \text{where } (u,v) \in E_x \iff u \in T_v(x) \wedge v \in T_u(x).$$

The **search task**: (1) output v^* if $\deg_{G_x}(v^*) \neq 1$, or (2) output a vertex $w \in V$ such that $w \neq v^*$ and $\deg_{G_x}(w)$ is odd.

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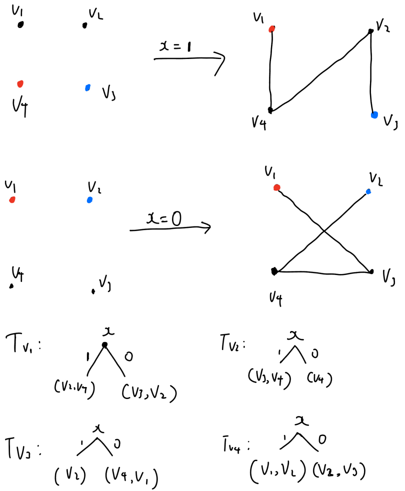
- On tt x, an END-OF-UNDIRECTED-LINE graph, G_x, is defined.
- This G_x is promise-true instance, solution to which is either (1) refuting its promise, or (2) finding an PPA-style solution.

Bridging TFNP^{dt} with Low-level Proof Systems

Query Complexity: Query PPA class, PPA^{dt} — one-bit, four-vertex example



Bridging $\mathsf{TFNP}^{\mathsf{dt}}$ with Low-level Proof Systems

Query Complexity: Query PPA class, PPA^{dt} — low-depth DT reduction

Class definition: A total search problem belongs to $\mathsf{PPA}^{\mathsf{dt}}$ iff it can be reduced to $\mathsf{PPA}^{\mathsf{dt}}$-OddDeg by the following reduction:

Definition (Low-depth decision tree reduction)

Let S be a $\mathsf{TFNP}^{\mathsf{dt}}$ problem with inputs $x \in \{0,1\}^N$. A *decision-tree reduction* from S to $\mathsf{PPA}^{\mathsf{dt}}$-OddDeg consists of:

1. **Graph constructor.** A family $\{T_v\}_{v \in V}$ of polylog(N)-depth decision trees (recall general $\mathsf{TFNP}^{\mathsf{dt}}$ definition) whose leaves describe the PPA graph G_x on poly(N) vertices.
2. **Solution translator.** A polylog(N)-height decision tree R that, on input (x, w) where w is a valid odd-degree vertex of G_x, outputs a witness $y \in S(x)$.

The reduction is *efficient* if every tree above has depth $O(\text{polylog}(N))$.

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Soundness. Low-depth DT that solves $\mathsf{PPA}^{\mathsf{dt}}$-OddDeg + reduction $\Longrightarrow$ low-depth decision tree that solves $S \in \mathsf{PPA}^{\mathsf{dt}}$.

Definition (Cost of $\mathcal{T} \in \mathsf{PPA}^{dt}$)

Take any PPA^{dt} instance S, it can be WLOG described by the complete problem $\mathcal{T} = (V, v^* \in V, \{T_v\}_{v \in V})$ on input $x \in \{0,1\}^N$, its cost is said to be

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Definition ($\mathsf{PPA}^{dt}(S)$)

$$\mathsf{PPA}^{dt}(S) = \min_{\mathcal{T} \text{ that solves } S} \{\text{cost of } \mathcal{T}\}.$$

Definition ($\mathbb{F}$-Nullstellensatz)

Let $\mathbb{F}$ be a field. An $\mathbb{F}$-Nullstellensatz instance is defined over the polynomial ring of the field $\mathbb{F}[z_1, z_2, \ldots, z_n]$, as such:

- For $i \in [m]$, $p_i \in \mathbb{F}[z_1, z_2, \ldots, z_n]$.
- $P := \{p_i = 0 : i \in [m]\}$ is an unsatisfiable system of polynomial equations.

Goal is to find a $\mathbb{F}$-Nullstellensatz refutation of P, which is a sequence of polynomials in the same polynomial ring:
$q_1, q_2, \ldots, q_m \in \mathbb{F}[z_1, z_2 \ldots, z_n]$, such that $\sum_{i\in[m]} q_i p_i = 1$, syntactically.

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Definition ($\mathbb{F}$-Nullstellensatz degree, $\mathsf{NS}_{\mathbb{F}}(P)$)

$$\mathsf{NS}_{\mathbb{F}}(P) = \min_{(q_1, \ldots, q_m)} \max_{q_i} \{\deg(q_i)\}.$$

Definition ($\mathsf{PPA}^{dt}(S)$)

$$\mathsf{PPA}^{dt}(S) = \min_{\mathcal{T} \text{ that solves } S} \max_{v \in V, x \in \{0,1\}^n} \{\text{height of } \mathcal{T}_v(x)\}.$$

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Bridging $\mathsf{TFNP}^{\mathsf{dt}}$ with Low-level Proof Systems

Landscape

Many equivalences have been set up between query complexity and low-level proof system complexity (summarized in [BFI23]):

- $\mathsf{FP}^{dt} \cong \mathsf{TreeRes}$ [LNNW95].
- $\mathsf{PLS}^{dt} \cong \mathsf{Res}$ [BKT14].
- $\mathsf{PPA}^{dt} \cong \mathbb{F}_2$-NS (given in this presentation as an example) [GKRS19].
- $\mathsf{PPA}_q^{dt} \cong \mathbb{F}_q$-NS, where q is a prime [Kam19].
- $\mathsf{PPADS}^{dt} \cong$ unary-NS [GHJ+24].
- $\mathsf{PPAD}^{dt} \cong$ unary-SA [GHJ+24].
- $\mathsf{SOPL}^{dt} \cong \mathsf{RevRes}$ [GHJ+24].
- $\mathsf{EOPL}^{dt} \cong \mathsf{RevResT}$ [GHJ+24].

Theorem (Main)

Let F be an unsatisfiable k-CNF. Then, $\mathsf{NS}_{\mathbb{F}_2}(F) = \Theta(\mathit{PPA}^{dt}(S(F)))$.

$\mathsf{PPA}^{\mathsf{dt}} \cong \mathbb{F}_2$-NS

Theorem stated and agenda for proofs

Theorem (Main)

Let F be an unsatisfiable k-CNF. Then, $\mathsf{NS}_{\mathbb{F}_2}(F) = \Theta(\mathsf{PPA}^{dt}(S(F)))$.

We prove the main theorem by showing the following directions:

Lemma (Stage ①; [BCE+95])

Let F be an unsatisfiable k-CNF. Then, $\mathsf{NS}_{\mathbb{F}_2}(F) \leq O(\mathsf{PPA}^{dt}(S(F)))$.

Lemma (Stage ②; (Theorem 4) of [GKRS19])

Let F be an unsatisfiable k-CNF. Then, $\mathsf{NS}_{\mathbb{F}_2}(F) \geq \Omega(\mathsf{PPA}^{dt}(S(F)))$.

A common framework for proving equivalences between query problems and low-level proof systems is using the depth of trees that solve FCSP as the translation layer. We will illustrate this framework in the two directions we set out to prove.

$\mathsf{PPA}^{\mathsf{dt}} \cong \mathbb{F}_2$-NS

A common framework

A common framework for proving equivalences between query problems and low-level proof systems is using the depth of trees that solve FCSP as the translation layer. We will illustrate this framework in the two directions we set out to prove.

Definition (False-Clause Search Problem)

Given an unsatisfiable k-CNF, $F = \bigwedge_{i \in [m]} C_i$, and also given an α as an assignment to F, find i such that $C_i(x) = 0$.

Direction 1: NS $\leq$ PPA^{dt}

Overview

Lemma

Let F be an unsatisfiable k-CNF. Then, $\mathsf{NS}_{\mathbb{F}_2}(F) \leq O(\mathsf{PPA}^{dt}(S(F)))$.

On a high level, NS can simulate PPA^{dt} effectively, in the sense that there is a low-degree family of polynomials that simulate a low-height tree that solves a PPA$^{\mathsf{dt}}$ problem.

Proof.

[BCE+95]. In the extended presentation in the uploaded recording. ∎

Lemma

Let F be an unsatisfiable k-CNF. Then, $\mathsf{NS}_{\mathbb{F}_2}(F) \geq \Omega(\mathsf{PPA}^{dt}(S(F)))$. Equivalently, $\mathsf{PPA}^{dt}(S(F)) \leq O(\mathsf{NS}_{\mathbb{F}_2}(F))$.

- Similarly, fix $F = \bigwedge_{i\in[m]} C_i$, and let p_i be the natural polynomial encoding of C_i.
- Fix a degree-d $\mathbb{F}_2$-Nullstellensatz refutation of F: $\sum_{i\in[m]} p_i q_i = 1$.
- It suffices to construct a cost-d PPA-decision tree $\mathcal{T} := (V, v^*, \{o_v\}_{v\in V}, \{\mathcal{T}_v\}_{v\in V})$ solving $S(F)$ (from the given refutation $\sum_{i\in[m]} p_i q_i = 1$).

- Expanding $\sum_{i\in[m]} p_i q_i$ into the sum of monomials.

Direction 2: $\mathsf{PPA}^{dt} \leq \mathsf{NS}$

Constructing $\mathcal{T} = (\underline{V, v^*, \{o_v\}}, \{\mathcal{T}_v\})$

- Expanding $\sum_{i \in [m]} p_i q_i$ into the sum of monomials.
- Each monomial corresponds to a unique vertex in V
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- $\forall i$, for every $v \in V_i$, let $o_v = C_i$ be its associated solution.

Direction 2: $\mathsf{PPA}^{dt} \leq \mathsf{NS}$

Constructing $\mathcal{T} = (V, v^*, \{o_v\}, \{\mathcal{T}_v\})$

Example

Let $F = (x_1 \vee x_2) \wedge (\neg x_1) \wedge (\neg x_2)$.

- The polynomial equations corresponding to F are $p_1 = (1 + x_1)(1 + x_2), p_2 = x_1, p_3 = x_2$.
- One $\mathbb{F}_2$-Nullstellensatz refutation is $q_1 = 1, q_2 = 1 + x_2, q_3 = 1$.
- $\sum_{i=1}^{3} p_i q_i = (1 + x_1 + x_2 + x_1 x_2) + (x_1 + x_1 x_2) + (x_2) = 1$.

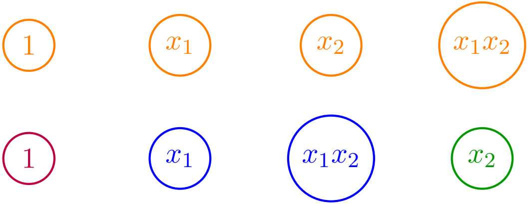

Figure: Vertices of $\mathcal{T}$

Direction 2: $\mathsf{PPA}^{dt} \leq \mathsf{NS}$

Constructing $\mathcal{T} = (V, v^*, \{o_v\}, \{\mathcal{T}_v\})$

- Since the polynomials $q_i p_i$ come from a valid $\mathbb{F}_2$-Nullstellensatz refutation, each monomial occurs an even number of times globally in the construction of V.
- Fix a global perfect matching M.
- Note that every edge in M are from two different subsets V_i and V_j ($i \neq j$, or V^*).

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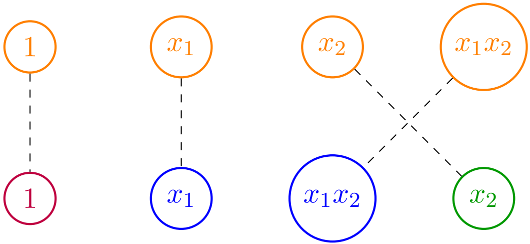

Figure: A global matching of V

On input x, constructing the edges of G_x (the vertex set is V):

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Constructing $\mathcal{T} = (V, v^*, \{o_v\}, \underline{\{\mathcal{T}_v\}})$

On input x, constructing the edges of G_x (the vertex set is V):

- (Out-group edges) For each $e \in M$ corresponding to a monomial $m = (v_i, v_j)$, add e_m to G_x if $m(x) = 1$.
- By adding e_m to G_x we mean including v_j in $\mathcal{T}_{v_i}(x)$ and v_i in $\mathcal{T}_{v_j}(x)$.

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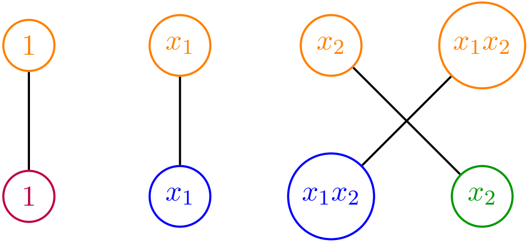

Figure: Out-group edges when $(x_1, x_2) = (1, 1)$

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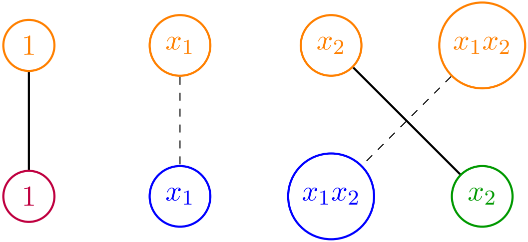

Figure: Out-group edges when $(x_1, x_2) = (0, 1)$

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On input x, constructing the edges of G_x (the vertex set is V):

- (<u>In-group edges</u>) Consider group V_i.
- If $C_i(x) = 0$, i.e., C_i is a valid solution, don't add any edges in V_i.
- Otherwise, $C_i(x) = 1$ and $p_i(x) = 0$.

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- Let $\rho := x \restriction \mathrm{vars}(p_i)$, and T_ρ be the multiset of *non-zero* monomials by applying ρ to each monomial in V_i (i.e., from $p_i \cdot q_i$).
- $p_i = 0 \implies p_i q_i = 0$. Therefore, each monomial $m' = m(\rho)$ in T_ρ occurs an even number of times.

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- $p_i = 0 \implies p_i q_i = 0$. Therefore, each monomial $m' = m(\rho)$ in T_ρ occurs an even number of times.
- Fix another perfect matching M_ρ.
- For each edge $e \in M_\rho$ corresponding to a monomial $m' := m(\rho)$, add e to G_x if $m(x) = 1 (= m'(x))$.

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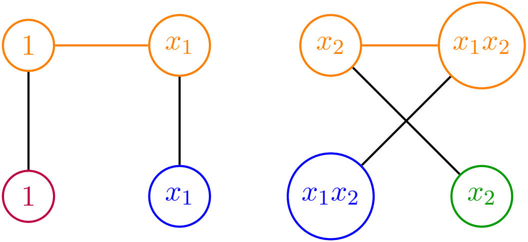

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- Complexity. For a vertex $v \in V_i$, its incident edge (of both types) can be decided by *the variables in* p_i plus *the variables in its corresponding monomial.* Therefore, height of $\mathcal{T}_v \leq d + k = O(d)$.

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- Complexity. For a vertex $v \in V_i$, its incident edge (of both types) can be decided by *the variables in p_i* plus *the variables in its corresponding monomial*. Therefore, height of $\mathcal{T}_v \leq d + k = O(d)$.
- Correctness. Fix x. $\deg(v^*) = 1$ always holds.
  For a monomial m in a true clause, let v be its corresponding vertex.
  - If $m(x) = 0$, $\deg(v) = 0$.
  - Otherwise, $m(x) \neq 0 \implies m(\rho) \neq 0$. The vertex corresponding to m will have both an out- and in-group neighbor. $\deg(v) = 2$.

Conclusion

- TFNP^{dt}, PPA^{dt} and the decision tree reduction;
- $\mathsf{NS}_{\mathbb{F}_2}(F) = \Theta(\mathsf{PPA}^{dt}(S(F)))$ for any unsatisfiable k-CNF F.
- A step of the lifting paradigm:

$$\text{Proof-complexity LB} \Longrightarrow \text{Query-complexity LB}$$

$$\overset{\text{lifting}}{\Longrightarrow} \text{Comm. LB} \overset{\text{KW}}{\Longrightarrow} \text{Model of Computation LB},$$

which further implies the lower bound of Monotone Span Programs / Linear SSS.

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